



NASA Technical Memorandum 81693

NASA-TM-81693 19820008539

Basic Lubrication Equations

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December 1981

NASA

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National Aeronautics and
Space Administration

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1981

N82-16413 #

The subject of elastohydrodynamic lubrication is identified with situations in which elastic deformation plays a significant role in the hydrodynamic lubrication process. This chapter is concerned with the basic equations used in the analysis of hydrodynamic lubrication. Basic elasticity theory is also considered in this chapter, and elastohydrodynamic theory is developed in Chapter 7.

Lubricants are usually Newtonian fluids, in which the rate of shear is linearly related to the shear stress. (A notable exception is grease, which behaves as a solid at shear stresses that are less than a threshold value and as a viscous fluid at higher stress levels.) The fluid is normally assumed to experience laminar flow. Navier (1823) derived the equations of fluid motion for these conditions from molecular considerations and from the introduction of Newton's hypothesis for a viscous fluid. Stokes (1845) also derived the governing equations of motion for a viscous fluid in a slightly different form, and the basic equations are thus known as the Navier-Stokes equations of

motion. The Navier-Stokes equations will be derived as simply as possible.

The study of hydrodynamic lubrication is, from a mathematical standpoint, the application of a reduced form of these Navier-Stokes equations in association with the continuity equation. The resulting differential equation was formulated by Reynolds (1886) in the wake of a classical experiment by Tower (1883) in which the existence of a thin fluid film was detected from measurements of pressures within the lubricant. Petrov (1883) had simultaneously recognized the existence of a coherent fluid film between the rotating shaft and stationary bearing in an extensive investigation of the friction of journal bearings. The Reynolds equation can be derived either from the Navier-Stokes and continuity equations or from first principles, provided of course that the same basic assumptions are adopted in each case. Both methods will be used in deriving the Reynolds equation, and the assumptions inherent in reducing the Navier-Stokes equations will be specified.

The Reynolds equation contains viscosity and density terms. These properties of the lubricant depend on temperature and pressure, and hence it is often necessary to couple the Reynolds equation with the energy equation. This chapter therefore deals with these lubricant properties and the energy equation.

The Reynolds equation also contains the film thickness as a parameter. The film thickness is a function of the elastic behavior of the bearing surface, and the governing elasticity equation is therefore presented in this chapter. The coupling of the Reynolds equation with the elasticity equation is considered in Chapter 7, where the basic concepts of elastohydrodynamic lubrication theory are developed.

5.1 Navier-Stokes Equations

The Navier-Stokes equations can be derived from a consideration of the dynamic equilibrium of an element of fluid. It is necessary to consider

- (1) Surface forces
- (2) Body forces
- (3) Inertia

5.1.1 Surface Forces

Figure 5.1 shows the stresses on the surfaces of a fluid element in a viscous fluid. Across each of the three mutually perpendicular surfaces there are three stresses, yielding a total of nine stress components. Of the three stresses acting on a given surface the normal stress is denoted by σ and the shear stress by τ . To avoid overcrowding, the stresses on the

surface perpendicular to the z axis have been omitted. The first subscript on the shear stresses refers to the coordinate direction perpendicular to the plane in which the stress acts, and the second designates the coordinate direction in which the stress acts. The following five relationships should be noted in relation to surface stresses:

(1) For equilibrium of the moments acting on the fluid element, the stresses must be symmetric; that is, the subscripts on the shear stresses can be reversed in order.

$$\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy} \quad (5.1)$$

(2) The hydrostatic pressure p in the fluid is considered to be the average of the three normal stress components.

$$\sigma_x + \sigma_y + \sigma_z = -3p \quad (5.2)$$

The minus sign is used because hydrostatic pressures are compressive, whereas positive stresses are tensile.

(3) The magnitude of the shear stresses depends on the rate at which the fluid is being distorted. For most fluids the dependence is of the form

$$\tau_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (5.3)$$

where

η = a constant of proportionality known as the coefficient of
absolute viscosity

u = components of velocity vector

The terms in parentheses in equation (5.3) are a measure of the distortion of the fluid element.

(4) The magnitude of the normal stresses can be written as

$$\sigma_i = -p + \lambda_a \xi + 2\eta \frac{\partial u_i}{\partial x_i} \quad (5.4)$$

where

$$\xi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (5.5)$$

and

λ_a = a second coefficient of viscosity

The divergence of the velocity vector, the dilatation, ξ measures the rate at which fluid is flowing out from each point; that is, it measures the expansion of the fluid. The factor 2 is introduced in equation (5.4) as a consequence of the definition of the coefficient of viscosity for a Newtonian fluid. A Newtonian fluid is one in which the shear stress is directly proportional to the rate of shear. This point is discussed in detail in texts on hydrodynamics where the relationship between shear and rate of strain components is considered in the direction of the principal axis (e.g., Lamb, 1932).

(5) By making use of equation (5.4) the following can be written

$$\sigma_x + \sigma_y + \sigma_z = -3p + (3\lambda_a + 2\eta)\xi \quad (5.6)$$

This equation shows that the average of the normal stresses differs from that previously described in equation (5.2). For these expressions to be compatible, the following must apply:

$$\lambda_a = -\frac{2}{3} \eta \quad (5.7)$$

The forces due to the stress gradients must be added to the external body forces. Three stresses tend to move the element in the x direction. Thus the change of τ_{xy} , for example, across the element and through a distance dy is $(\partial\tau_{xy}/\partial y)dy$. This stress acts on the face of the fluid element with area $dx dz$ and produces a force $(\partial\tau_{xy}/\partial y)dx dy dz$. There are similar expressions for σ_x and τ_{xz} .

5.1.2 Body Forces

The forces needed to accelerate an element of fluid may be supplied in part by an external force field, perhaps gravity, associated with the whole body of the element. If the components of the external force field per unit mass are X_a , Y_a , and Z_a , these forces acting on an element are

$$X_a \rho dx dy dz, \quad Y_a \rho dx dy dz, \quad Z_a \rho dx dy dz \quad (5.8)$$

5.1.3 Inertia

The three components of acceleration of the fluid are the three total derivatives Du/Dt , Dv/Dt , and Dw/Dt . The significance of the total derivatives can be seen from the following. Consider the x component of velocity u . In general, $u = f(x, y, z, t)$. The change in u that occurs in time dt is approximately

$$Du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \quad (5.9)$$

In the limit as $dt \rightarrow 0$, $dx/dt = u$, $dy/dt = v$, and $dz/dt = w$. Therefore, if equation (5.9) is divided throughout by dt , while making use of the above, we can write the total derivative for the u component as

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \quad (5.10)$$

Similarly for the v and w components of velocity we can write

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \quad (5.11)$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \quad (5.12)$$

The total time derivative measures the change in velocity of one element of fluid as it moves about in space. The term $\partial/\partial t$ is known as the local derivative since it gives the variation of

velocity with time at a fixed point. The last three terms are grouped together under the heading "convective differential."

The mass of an element of fluid having dimensions dx , dy , and dz is $\rho \, dx \, dy \, dz$; thus the components of the resultant forces required to accelerate the element are

$$\rho \frac{Du}{Dt} \, dx \, dy \, dz, \quad \rho \frac{Dv}{Dt} \, dx \, dy \, dz, \quad \rho \frac{Dw}{Dt} \, dx \, dy \, dz \quad (5.13)$$

5.1.4 Equilibrium

Having defined the body, surface, and inertia forces acting on a fluid element, we can now state the requirement for dynamic equilibrium mathematically. When the common factor $dx \, dy \, dz$ is eliminated from each term and the resulting surface and body forces are related to the acceleration of an element of fluid, it is found that

$$\rho \frac{Du}{Dt} = \rho X_a + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \quad (5.14)$$

Also there are similar expressions for stress gradients and body forces that tend to move the element in the y and z directions.

$$\rho \frac{Dv}{Dt} = \rho Y_a + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \quad (5.15)$$

$$\rho \frac{Dw}{Dt} = \rho Z_a + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \quad (5.16)$$

The Navier-Stokes equations or the equations of motion can be obtained in terms of the velocity derivatives by substituting the results of equations (5.3) to (5.7) into equations (5.14) to (5.16). The resulting equations are

$$\left. \begin{aligned} \rho \frac{Du}{Dt} &= \rho X_a - \frac{\partial p}{\partial x} + \frac{2}{3} \frac{\partial}{\partial x} \left[\eta \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \right] + \frac{2}{3} \frac{\partial}{\partial x} \left[\eta \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) \right] \\ &\quad + \frac{\partial}{\partial y} \left[\eta \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[\eta \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] \\ \rho \frac{Dv}{Dt} &= \rho Y_a - \frac{\partial p}{\partial y} + \frac{2}{3} \frac{\partial}{\partial y} \left[\eta \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \right] + \frac{2}{3} \frac{\partial}{\partial y} \left[\eta \left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right) \right] \\ &\quad + \frac{\partial}{\partial x} \left[\eta \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial z} \left[\eta \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \\ \rho \frac{Dw}{Dt} &= \rho Z_a - \frac{\partial p}{\partial z} + \frac{2}{3} \frac{\partial}{\partial z} \left[\eta \left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} \right) \right] + \frac{2}{3} \frac{\partial}{\partial z} \left[\eta \left(\frac{\partial w}{\partial z} - \frac{\partial v}{\partial y} \right) \right] \\ &\quad + \frac{\partial}{\partial y} \left[\eta \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \right] + \frac{\partial}{\partial x} \left[\eta \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \end{aligned} \right\} \quad (5.17)$$

The terms on the left side of these equations represent inertia effects, and those on the right side are the body forces, pressure gradient, and viscous terms in that order. Equations (5.17) are the most general form of the Navier-Stokes equations for a Newtonian fluid.

For an incompressible fluid these equations reduce to

$$\left. \begin{aligned} \rho \frac{Du}{Dt} &= \rho X_a - \frac{\partial p}{\partial x} + \eta \nabla^2 u \\ \rho \frac{Dv}{Dt} &= \rho Y_a - \frac{\partial p}{\partial y} + \eta \nabla^2 v \\ \rho \frac{Dw}{Dt} &= \rho Z_a - \frac{\partial p}{\partial z} + \eta \nabla^2 w \end{aligned} \right\} \quad (5.18)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (5.19)$$

In cylindrical polar coordinates with r_c , ϕ_c , and z such that $x = r_c \cos \phi_c$ and $y = r_c \sin \phi_c$, equations (5.17) can be rewritten as

$$\left. \begin{aligned} \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r_c} + \frac{v}{r_c} \frac{\partial u}{\partial \phi_c} + w \frac{\partial u}{\partial z} - \frac{v^2}{r_c} \right) &= \rho X_c - \frac{\partial p}{\partial r_c} \\ &+ \eta \left(\nabla^2 u - \frac{u}{r_c^2} - \frac{2}{r_c^2} \frac{\partial v}{\partial \phi_c} \right) \\ \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r_c} + \frac{v}{r_c} \frac{\partial v}{\partial \phi_c} + w \frac{\partial v}{\partial z} + \frac{uv}{r_c} \right) &= \rho Y_c - \frac{1}{r_c} \frac{\partial p}{\partial \phi_c} \\ &+ \eta \left(\nabla^2 v - \frac{v}{r_c^2} + \frac{2}{r_c^2} \frac{\partial u}{\partial \phi_c} \right) \\ \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r_c} + \frac{v}{r_c} \frac{\partial w}{\partial \phi_c} + w \frac{\partial w}{\partial z} \right) &= \rho Z_c - \frac{\partial p}{\partial z} + \eta \nabla^2 w \end{aligned} \right\} \quad (5.20)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r_c^2} + \frac{1}{r_c} \frac{\partial}{\partial r_c} + \frac{1}{r_c^2} \frac{\partial^2}{\partial \phi_c^2} + \frac{\partial^2}{\partial z^2} \quad (5.21)$$

In spherical polar coordinates with r_s , θ_s , and ϕ_s such that $x = r_s \sin \theta_s \cos \phi_s$, $y = r_s \sin \theta_s \sin \phi_s$, and $z = r_s \cos \theta_s$, the Navier-Stokes equations for an incompressible fluid are

$$\left. \begin{aligned} & \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r_s} + \frac{v}{r_s} \frac{\partial u}{\partial \theta_s} + \frac{w}{r_s \sin \theta_s} \frac{\partial u}{\partial \phi_s} - \frac{v^2 + w^2}{r_s} \right) \\ &= \rho X_s + \frac{\partial p}{\partial r_s} + \eta \left(\nabla^2 u - \frac{2u}{r_s^2} - \frac{2}{r_s^2} \frac{\partial v}{\partial \theta_s} - \frac{2v \cot \theta_s}{r_s^2} - \frac{2}{r_s^2 \sin \theta_s} \frac{\partial w}{\partial \phi_s} \right) \\ & \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r_s} + \frac{v}{r_s} \frac{\partial v}{\partial \theta_s} + \frac{w}{r_s \sin \theta_s} \frac{\partial v}{\partial \phi_s} + \frac{uv}{r_s} - \frac{w^2 \cot \theta_s}{r_s} \right) \\ &= \rho Y_s - \frac{1}{r_s} \frac{\partial p}{\partial \theta_s} + \eta \left(\nabla^2 v + \frac{2}{r_s^2} \frac{\partial u}{\partial \theta_s} - \frac{v}{r_s^2 \sin^2 \theta_s} - \frac{2 \cos \theta_s}{r_s^2 \sin^2 \theta_s} \frac{\partial w}{\partial \phi_s} \right) \\ & \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r_s} + \frac{v}{r_s} \frac{\partial w}{\partial \theta_s} + \frac{w}{r_s \sin \theta_s} \frac{\partial w}{\partial \phi_s} + \frac{wu}{r_s} + \frac{vw \cot \theta_s}{r_s} \right) \\ &= \rho Z_s - \frac{1}{r_s \sin \theta_s} \frac{\partial p}{\partial \phi_s} + \eta \left(\nabla^2 w - \frac{w}{r_s^2 \sin^2 \theta_s} + \frac{2}{r_s^2 \sin \theta_s} \frac{\partial u}{\partial \phi_s} \right. \\ & \quad \left. + \frac{2 \cos \theta_s}{r_s^2 \sin^2 \theta_s} \frac{\partial v}{\partial \phi_s} \right) \end{aligned} \right\} \quad (5.22)$$

where

$$\nabla^2 = \frac{1}{r_s^2} \frac{\partial}{\partial r_s} \left(r_s^2 \frac{\partial}{\partial r_s} \right) + \frac{1}{r_s^2 \sin \theta_s} \frac{\partial}{\partial \theta_s} \left(\sin \theta_s \frac{\partial}{\partial \theta_s} \right) + \frac{1}{r_s^2 \sin^2 \theta_s} \frac{\partial^2}{\partial \phi_s^2} \quad (5.23)$$

5.2 Continuity Equation

The Navier-Stokes equations contain four unknowns: u , v , w , and p . The viscosity and density can be written as functions of pressure and temperature. A fourth equation is supplied by the continuity equation.

In continuous motion the increase in the mass of fluid within a fixed control volume in time dt must be equal to the excess of mass that flows in over the mass that flows out. Consider the mass flow across a fluid element such as that shown in Figure 5.1. The mass flowing into the element in the x direction in time dt is $\{\rho u + [\partial(\rho u)/\partial x]dx\}dy dz dt$; thus the excess of mass flow into the element in the x direction is $-[\partial(\rho u)/\partial x]dx dy dz dt$. Similarly, when the directions y and z are considered, the total excess of mass flow into the element in time dt is

$$- \left[\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right] dx dy dz dt$$

Now the original mass inside the element is $\rho \, dx \, dy \, dz$ and the increase in mass inside the element in time dt is $(\partial\rho/\partial t)dx \, dy \, dz \, dt$. Thus

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \quad (5.24)$$

This then is the continuity equation in Cartesian coordinates. If the flow is steady, $\partial\rho/\partial t = 0$. Also if the fluid is incompressible, ρ is constant and the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (5.25)$$

5.3 Reynolds Equation

The differential equation governing the pressure distribution in a fluid-film bearing is known as the Reynolds equation. As pointed out earlier, this equation was first derived in a remarkable paper by Reynolds (1886). Reynolds' classical paper contained not only the basic differential equation of fluid-film lubrication, but also a direct comparison between his theoretical predictions and the experimental results obtained by Tower (1883). Reynolds, however, restricted his analysis to an incompressible fluid. This is an unnecessary restriction, and Harrison (1913) included effects of compressibility and dynamic loading in a later analysis. In this section a generalized Reynolds equation is derived from the Navier-Stokes and contin-

uity equations after assumptions that are normally valid for fluid-film lubrication have been introduced. The Reynolds equation is also derived directly from the principle of mass conservation and simple expressions for Couette and Poiseuille flow. The various terms in the Reynolds equation are discussed, and various forms of the Reynolds equation as they apply to particular cases of bearing operation are presented at the end of this chapter.

5.3.1 Derivation of Reynolds Equation From the Navier-Stokes and Continuity Equations

The basic equation of fluid-film lubrication, the Reynolds equation, can be derived from the reduced form of the Navier-Stokes and continuity equations. Recall that a Newtonian fluid was assumed in the derivation of the Navier-Stokes equations. A minimum number of restrictive assumptions are introduced that enable a general form of the Reynolds equation to be established. The assumptions are

(1) Inertia and body force terms are negligible compared with the pressure and viscous terms.

(2) There is a negligible variation of pressure across the fluid film ($\partial p / \partial z = 0$).

(3) Owing to the geometry of the fluid film the derivatives of u and v with respect to z are much larger than the other derivatives of velocity components.

The last two assumptions arise from the geometrical form of the lubricating film. If h and l represent typical dimensions of film thickness and bearing length, respectively, it is found that the ratio h/l rarely exceeds 10^{-2} , is often as small as 10^{-4} , and is most commonly about 10^{-3} . The lubricating film can thus be seen as a long, thin channel of small taper in all realistic bearing configurations.

The application of these assumptions allows the Navier-Stokes equations expressed in equations (5.17) to be reduced to

$$\left. \begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial}{\partial z} \left(\eta \frac{\partial u}{\partial z} \right) \\ \frac{\partial p}{\partial y} &= \frac{\partial}{\partial z} \left(\eta \frac{\partial v}{\partial z} \right) \end{aligned} \right\} \quad (5.26)$$

Since the pressure has been assumed to be a function of x and y only, these equations can be integrated directly to give general expressions for the velocity gradients:

$$\left. \begin{aligned} \frac{\partial u}{\partial z} &= \frac{z}{\eta} \frac{\partial p}{\partial x} + \frac{A}{\eta} \\ \frac{\partial v}{\partial z} &= \frac{z}{\eta} \frac{\partial p}{\partial y} + \frac{C}{\eta} \end{aligned} \right\} \quad (5.27)$$

where A and C are constants.

Now the viscosity of the lubricant may change considerably across the thin film (z direction) as a result of temperature variations that arise in some bearing problems. In this case, progress toward a satisfactory Reynolds equation is considerably complicated.

An approach that is satisfactory in the majority of fluid-film applications is to treat η as the average value of the viscosity across the film. Note that this does not restrict the variation of viscosity in the x and y directions. This approach is pursued here, but a more general derivation of the Reynolds equation that takes account of fluid property variations along and across the film and that discusses in detail the assumptions made earlier has been presented by Dowson (1962).

With η representing an average value of viscosity across the film, the velocity components become

$$\left. \begin{aligned} u &= \frac{z^2}{2\eta} \frac{\partial p}{\partial x} + A \frac{z}{\eta} + B \\ v &= \frac{z^2}{2\eta} \frac{\partial p}{\partial y} + C \frac{z}{\eta} + D \end{aligned} \right\} \quad (5.28)$$

If we assume zero slip at the fluid-solid interface, the boundary values for velocity are

$$\left. \begin{aligned} z = 0, \quad u &= u_b, \quad v = v_b \\ z = h, \quad u &= u_a, \quad v = v_a \end{aligned} \right\} \quad (5.29)$$

The subscripts a and b refer to conditions on the upper (curved) and lower (plane) surfaces, respectively. Therefore u_a , v_a , and w_a refer to the velocity components in the x, y, and z directions of the upper surface and u_b , v_b , and w_b refer to the velocity components of the lower surface in the same directions.

With the boundary values given in equation (5.29), the velocity gradients and components can be written as

$$\left. \begin{aligned} \frac{\partial u}{\partial z} &= \left(\frac{2z - h}{2\eta} \right) \frac{\partial p}{\partial x} - \frac{u_b - u_a}{h} \\ \frac{\partial v}{\partial z} &= \left(\frac{2z - h}{2\eta} \right) \frac{\partial p}{\partial y} - \frac{v_b - v_a}{h} \end{aligned} \right\} \quad (5.30)$$

$$\left. \begin{aligned} u &= -z \left(\frac{h - z}{2\eta} \right) \frac{\partial p}{\partial x} + u_b \left(\frac{h - z}{h} \right) + u_a \frac{z}{h} \\ v &= -z \left(\frac{h - z}{2\eta} \right) \frac{\partial p}{\partial y} + v_b \left(\frac{h - z}{h} \right) + v_a \frac{z}{h} \end{aligned} \right\} \quad (5.31)$$

With these expressions for the velocity gradients and the velocity components we can now derive expressions for the surface stresses and the volume flow rate.

Surface Stresses

The viscous shear stresses acting on the solids as defined in equation (5.3) are $(\eta \partial u / \partial z)_{z=0}$ and $-(\eta \partial u / \partial z)_{z=h}$. These expressions can be evaluated from equation (5.30) as

$$\left. \begin{aligned} \left(\eta \frac{\partial u}{\partial z} \right)_{z=0} &= -\frac{h}{2} \frac{\partial p}{\partial x} - \frac{\eta(u_b - u_a)}{h} \\ - \left(\eta \frac{\partial u}{\partial z} \right)_{z=h} &= -\frac{h}{2} \frac{\partial p}{\partial x} + \frac{\eta(u_b - u_a)}{h} \end{aligned} \right\} \quad (5.32)$$

Volume Flow Rate

The volume rates of flow per unit width in the x and y directions are defined as

$$q_x = \int_0^h u \, dz \quad \text{and} \quad q_y = \int_0^h v \, dz \quad (5.33)$$

Substituting equation (5.31) into equation (5.33) while integrating gives

$$\left. \begin{aligned} q_x &= -\frac{h^3}{12\eta} \frac{\partial p}{\partial x} + \left(\frac{u_b + u_a}{2} \right) h \\ q_y &= -\frac{h^3}{12\eta} \frac{\partial p}{\partial y} + \left(\frac{v_b + v_a}{2} \right) h \end{aligned} \right\} \quad (5.34)$$

The first term on the right side of these equations represents the well-known Poiseuille (or pressure) flow, and the second term represents the Couette (or velocity) flow.

Returning to equation (5.31), the Reynolds equation is formed by introducing these expressions into the continuity

equation (5.24). Before doing so, however, it is convenient to express the continuity equation in integral form.

$$\int_0^h \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right] dz = 0$$

Now a general rule of integration is that

$$\int_0^h \frac{\partial}{\partial x} [f(x, y, z)] dz = \frac{\partial}{\partial x} \int_0^h f(x, y, z) dz - f(x, y, h) \frac{\partial h}{\partial x}$$

Hence, if the density ρ is assumed to be the mean density across the film, the integrated continuity equation becomes

$$\begin{aligned} h \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left(\rho \int_0^h u dz \right) - \rho u_a \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} \left(\rho \int_0^h v dz \right) \\ - \rho v_a \frac{\partial h}{\partial y} + \rho (w_a - w_b) = 0 \quad (5.35) \end{aligned}$$

The integrals in this expression represent the volume rate of flow per unit width (q_x and q_y) described in equation (5.34). When these flow-rate expressions (which are derived from the Navier-Stokes equations) are introduced into the integrated continuity equation, we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \left(-\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(-\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial x} \left[\frac{\rho (u_b + u_a)h}{2} \right] + \frac{\partial}{\partial y} \left[\frac{\rho (v_b + v_a)h}{2} \right] \\ + \rho (w_a - w_b) - \rho u_a \frac{\partial h}{\partial x} - \rho v_a \frac{\partial h}{\partial y} + h \frac{\partial \rho}{\partial t} = 0 \quad (5.36) \end{aligned}$$

The first two terms are Poiseuille terms and describe the net flow rates due to pressure gradients within the lubricated area; the third and fourth terms are the Couette terms and describe the net entraining flow rates due to surface velocities. The fifth, sixth, and seventh terms describe the net flow rates due to a squeezing motion, and the last term describes the net flow rates due to local compression. The last four terms can be combined and written as $\partial(\rho h)/\partial t$. The generalized Reynolds equation then becomes

$$\frac{\partial}{\partial x} \left(\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial y} \right) = \frac{\partial}{\partial x} \left[\frac{\rho(u_a + u_b)h}{2} \right] + \frac{\partial}{\partial y} \left[\frac{\rho(v_a + v_b)h}{2} \right] + \frac{\partial(\rho h)}{\partial t} \quad (5.37)$$

Equations (5.36) and (5.37) represent the general form of the Reynolds equation within the range of assumptions discussed.

5.3.2 Direct Derivation of the Reynolds Equation From the Laws of Viscous Flow and the Principle of Mass Conservation

The Reynolds equation can be derived directly by considering a control column fixed in space and extending across the film. Before discussing the control volume, however, we note that three additional assumptions were introduced into the derivation outlined in the previous section. These additional assumptions were

(1) The flow is laminar. Turbulent flow is sometimes encountered in high-speed, fluid-film bearings, and a modified Reynolds equation is required for this condition.

(2) There is no slip between the fluid and bearing solids at common boundaries.

(3) The lubricant properties (viscosity and density) are constant across the thickness of the film.

If these three assumptions, along with those given at the beginning of the previous section, are used, the two approaches to the derivation of the Reynolds equation will be compatible.

Consider the rate of mass flow through a rectangular section control volume of sides $\Delta x, \Delta y$ fixed in the coordinate system and extending across the lubricant film between the two bearing surfaces as shown in Figure 5.2. Note that one bearing surface is represented by the plane $z = 0$ and the other by a curved surface such that the film thickness at any instant is a function of x and y only. This is exactly the coordinate system used in the previous derivation of the Reynolds equation.

The mass of lubricant in the control volume at any instant is $\rho h \Delta x \Delta y$. The rate of change within the control space arises from the change in both the density and the height of the column and is given by $[(\partial(\rho h)/\partial t)]\Delta x \Delta y$. The difference between the rate of mass flowing into the control space and the rate of mass leaving the control volume is $-(\partial q_x/\partial x)\Delta x \Delta y$ in the x direction and $-(\partial q_y/\partial y)\Delta x \Delta y$ in the y direction.

The principle of mass conservation demands that the rate at which mass is accumulating in the control space ($\partial(\rho h)/\partial t$) must be equal to the difference between the rates at which mass enters and the rate at which mass leaves. Therefore

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} = \frac{\partial}{\partial t} (\rho h) \quad (5.38)$$

But

$$\frac{\partial}{\partial t} (\rho h) = \rho \frac{\partial h}{\partial t} + h \frac{\partial \rho}{\partial t}$$

and

$$\frac{\partial}{\partial t} (\rho h) = \rho \left(w_a - w_b - u_a \frac{\partial h}{\partial x} - v_a \frac{\partial h}{\partial y} \right) + h \frac{\partial \rho}{\partial t} \quad (5.39)$$

By making use of equations (5.39) and (5.34), equation (5.38) can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial y} \right) &= \frac{\partial}{\partial x} \left[\frac{\rho h (u_b + u_a)}{2} \right] + \frac{\partial}{\partial y} \left[\frac{\rho h (v_b + v_a)}{2} \right] \\ &+ \rho \left(w_a - w_b - u_a \frac{\partial h}{\partial x} - v_a \frac{\partial h}{\partial y} \right) + h \frac{\partial \rho}{\partial t} \end{aligned} \quad (5.40)$$

This is a generalized Reynolds equation for the assumptions discussed and is exactly the same as that derived in the previous section (see equation (5.36)). The equation can of course be reduced to

$$\frac{\partial}{\partial x} \left(\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial y} \right) = \frac{\partial}{\partial x} \left[\frac{\rho (u_a + u_b) h}{2} \right] + \frac{\partial}{\partial y} \left[\frac{\rho (v_a + v_b) h}{2} \right] + \frac{\partial}{\partial t} (\rho h) \quad (5.41)$$

to be consistent with equation (5.37).

5.3.3 Standard Reduced Forms

The specification of boundary velocities u_b , v_b , and w_b for the plane surface does not present any difficulty. In general the surface velocities of the upper bearing components need to be related to the translation of the component parallel to the coordinate axes and the rotation of the component about its own center. In subsequent chapters we shall be concerned with boundary velocities confined to the following values:

$$\left. \begin{array}{lll} u_b = u_b, & v_b = 0, & w_b = 0 \\ u_a = u_a, & v_a = 0, & w_a = u_a \frac{\partial h}{\partial x} \end{array} \right\} \quad (5.42)$$

where u_b and u_a are the surface velocities of the two bearing components in the x direction. Note that the coordinate x was associated with the semiminor axis of elliptical contacts in Section 2.2.4. If the surface velocities do not lie in the direction of the semiminor axis, the coordinate x should always be taken as the direction of surface motion, even if k is then less than unity.

For steady-state conditions the Reynolds equation (as stated in equation (5.40)) becomes

$$\frac{\partial}{\partial x} \left(\frac{\rho h^3}{\eta} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\rho h^3}{\eta} \frac{\partial p}{\partial y} \right) = 12u \frac{\partial(\rho h)}{\partial x} \quad (5.43)$$

where

$$u = \frac{u_a + u_b}{2} \quad (5.44)$$

Again it should be emphasized that equation (5.43) allows for variation of the viscosity and density in the x and y directions. If, however, the fluid properties do not vary significantly, the corresponding Reynolds equation is

$$\frac{\partial}{\partial x} \left(h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(h^3 \frac{\partial p}{\partial y} \right) = 12\eta u \frac{\partial h}{\partial x} \quad (5.45)$$

Equation (5.43) not only allows the fluid properties to vary in the x and y directions, but also permits the bearing surfaces to be of finite length in the y direction. Side leakage, or flow in the y direction, is associated with the second term in equation (5.43). If the pressure in the lubricant has to be considered as a function of x and y , the solution of equation (5.43) can rarely be achieved analytically. Approximate numerical solutions are therefore sought as outlined in Chapter 7, where the elastohydrodynamic theory for elliptical contacts is developed.

In many conventional lubrication problems side leakage can be neglected, and this often leads to analytical solutions. If side leakage is neglected, equation (5.43) reduces to

$$\frac{\partial}{\partial x} \left(\frac{\rho h^3}{\eta} \frac{\partial p}{\partial x} \right) = 12u \frac{\partial(\rho h)}{\partial x} \quad (5.46)$$

This equation can be integrated with respect to x to yield the familiar integrated form of the Reynolds equation

$$\frac{dp}{dx} = 12\eta u \left[\frac{\rho h - (\rho h)_m}{\rho h^3} \right] \quad (5.47)$$

where the subscript m refers to the conditions at points where $dp/dx = 0$, such as the point of maximum pressure. If the lubricant is incompressible, equation (5.47) reduces further to

$$\frac{dp}{dx} = 12\eta u \left(\frac{h - h_m}{h^3} \right) \quad (5.48)$$

5.4 Viscosity

The viscosity of a fluid may be associated with its resistance to flow, with the resistance arising from intermolecular forces and internal friction as the molecules move past each other. Thick fluids, like molasses, have relatively high viscosity; they do not flow easily. Thinner fluids, like water, have lower viscosity; they flow very easily.

Simple kinetic theory gives a good molecular description of the phenomenon of viscosity in gases, but a physical description of the viscosity of liquids, where molecular activity is restricted, is less well developed.

The foundations of modern viscous flow theory were laid in the seventeenth century by Sir Isaac Newton, who proposed a method of quantifying the viscosity of a fluid. He considered the flow to be equivalent to a large number of thin layers of fluid sliding over each other. The internal friction or viscosity of the fluid was assumed to give rise to shear stresses τ between the sliding layers. These stresses act in such a way

that they tend to retard the faster moving layers and accelerate the slower layers. Newton (1687) postulated that the viscous shear stresses were directly proportional to the shear strain rate. The relationship for internal friction in a viscous fluid (as proposed by Newton, 1687) can be written as

$$\tau = \eta \frac{du}{dz} \quad (5.49)$$

where

τ = internal shear stress in the fluid

η = coefficient of absolute or dynamic viscosity or coefficient of internal friction

du/dz = velocity gradient across film

It follows from equation (5.49) that the unit of viscosity must be the unit of shear stress divided by the unit of shear rate. In the newton-meter-second system, the unit of shear stress is the newton per square meter. Hence, the unit of viscosity will be newton per square meter multiplied by second, or N s/m^2 . In the SI system the unit of pressure or stress (N/m^2) is known as a pascal, abbreviated Pa, and it is becoming increasingly common to refer to the SI unit of viscosity as the pascal second (Pa s). In the cgs system, where the dyne is the unit of force, viscosity is expressed as dyne-second per square centimeter. This unit is called the poise after Poiseuille who, though primarily interested in the movement of blood, studied the flow of distilled water in very narrow tubes. It happens

that a viscosity of 1 poise is rather high; hence the frequent use of the centipoise (cP).

Conversion of viscosity from one system to another can be facilitated by Table 5.1. To convert from a unit in the column on the left side of the table to a unit at the top of the table, multiply by the corresponding value given in the table. For example, $\eta = 0.04 \text{ N s/m}^2 = 0.04 \times 1.45 \times 10^{-4} \text{ lbf s/in}^2 = 5.8 \times 10^{-6} \text{ lbf s/in}^2$. Three metric and one English system are presented - all based on force, length, and time. Metric units are the centipoise, the kilogram-second per square meter, and the newton-second per square meter (or Pa s). The English unit is pound-second per square inch, or reyn, in honor of Osborne Reynolds.

Viscosity is the most important property of the lubricants employed in hydrodynamic and elastohydrodynamic lubrication. In general, however, the viscosity of a lubricant does not simply assume a uniform level in a given bearing. This results from the nonuniformity of the pressure and/or temperature prevailing in the lubricant film. Indeed many elastohydrodynamically lubricated machine elements operate over ranges of pressure and/or temperature so extensive that the consequent variations in the viscosity of the lubricant may become substantial and, in turn, may dominate the operating characteristics of the bearing.

Consequently an adequate knowledge of the viscosity-pressure and viscosity-pressure-temperature relationships of lubri-

cants is indispensable. The next two sections therefore deal with such relationships.

5.4.1 Effect of Pressure on Viscosity

As long ago as 1893, Barus proposed the following formula for the isothermal viscosity-pressure dependence of liquids:

$$^*\ln \left(\frac{\eta}{\eta_0} \right) = \alpha p \quad (5.50)$$

where

η = viscosity at gauge pressure

η_0 = viscosity at atmospheric pressure

α = pressure-viscosity coefficient of lubricant

The pressure-viscosity coefficient α characterizes the liquid considered and depends only on temperature, not on pressure. Although equation (5.50) is extensively used, it is not generally applicable and is valid as a reasonable approximation only in a moderate-pressure range.

Because of the shortcomings of equation (5.50), several isothermal viscosity-pressure formulas have been proposed that usually contain two or more parameters instead of the single parameter suggested by Barus (1893). One of these approaches, which is used in this book, was developed by Roelands (1966),

*log denotes the common or Briggsian logarithm, \log_{10} ;

ln denotes the natural or Napierian logarithm, \log_e .

who undertook a wide-ranging study of the effect of pressure on the viscosity of lubricants. For isothermal conditions the Roelands (1966) formula can be written as

$$\log \eta + 1.200 = \left(\log \eta_0 + 1.200 \right) \left(1 + \frac{p}{2000} \right)^{Z_1} \quad (5.51)$$

where

p = gauge pressure, kgf/cm^2

Z_1 = viscosity-pressure index, a dimensionless constant

By taking the antilog of both sides and rearranging terms, equation (5.51) becomes

$$\eta = \eta_0 (1+p/2000)^{Z_1} \times 10^{1.2[(1+p/2000)^{Z_1} - 1]}$$

By writing this in dimensionless terms and rearranging,

$$\frac{\eta}{\eta_0} = \left(\frac{\eta_\infty}{\eta_0} \right)^{[1 - (1+p/c)^{Z_1}]} \quad (5.52)$$

where

$$\eta_\infty = 6.31 \times 10^{-5} \text{ N s/m}^2 \quad (9.15 \times 10^{-9} \text{ lbf s/in}^2)$$

$$c = 1.96 \times 10^8 \text{ N/m}^2 \quad (28,440 \text{ lbf/in}^2)$$

In equations (5.51) and (5.52), care must be taken to ensure that the same dimensions are used in defining the constants.

In the Roelands (1966) formulation the lubricant is defined by the atmospheric viscosity η_0 , the viscosity-pressure index Z_1 , and the asymptotic isoviscous pressure $p_{iv,as}$.

The equation describing the latter quantity can be written as

$$p_{iv,as} = \eta_0 \int_0^\infty \frac{dp}{\eta} \quad (5.53)$$

Blok (1965) arrived at the very important conclusion that all elastohydrodynamic lubrication results achieved hitherto for an exponential-pressure dependence (see equation (5.50)) can, to a fair approximation, be generalized simply by substituting the reciprocal of the asymptotic isoviscous pressure $1/p_{iv,as}$ for the viscosity-pressure coefficient α occurring in those results. This implies that

$$\alpha \approx \frac{1}{p_{iv,as}} \quad (5.54)$$

5.4.2 Effects of Pressure and Temperature on Viscosity

The viscosity is found to be extremely sensitive to both pressure and temperature. This extreme sensitivity forms a considerable obstacle to the analytical description of the consequent viscosity changes. Roelands (1966) noted that at constant pressure the viscosity increases more or less exponentially with the reciprocal of absolute temperature. Similarly at constant temperature the viscosity increases more or less exponentially with pressure as shown earlier in this chapter. In general, however, the relevant exponential relationships constitute only first approximations and may be resorted to only in moderate-temperature ranges.

From Roelands (1966) the viscosity-temperature-pressure equation can be written as

$$\log(\log \eta + 1.200) = - S_0 \log \left(1 + \frac{T_m}{135} \right) + \left[D_2 - C_2 \log \left(1 + \frac{T_m}{135} \right) \right] \log \left(1 + \frac{p}{2000} \right) + \log G_0$$

Taking the antilog we get

$$\log \eta + 1.200 = G_0 \frac{\left(1 + \frac{p}{2000} \right)^{-C_2 \log(1+T_m/135) + D_2}}{\left(1 + \frac{T_m}{135} \right)^{S_0}} \quad (5.55)$$

According to equation (5.55) four parameters G_0 , S_0 , C_2 , and D_2 are sufficient to enable the viscosity η to be expressed in centipoise as a function of temperature T_m in degrees Celsius and gauge pressure p in kgf/cm^2 .

5.5 Density

The effects of temperature on viscosity were found in the preceding section to be most important. For a comparable change in pressure and/or temperature the density change is small compared with the viscosity change. However, very high pressures exist in elastohydrodynamic films, and the liquid can no longer be considered as an incompressible medium. It is therefore necessary to consider the dependence of the density on pressure.

The variation of density with pressure is roughly linear at low pressures, but the rate of increase falls away at high pressures. The limit of the compression of mineral oils is only

25 percent for a maximum density increase of about 33 percent. From Dowson and Higginson (1966) the dimensionless density for mineral oil can be written as

$$\bar{\rho} = \frac{\rho}{\rho_0} = 1 + \frac{0.6 p}{1 + 1.7 p} \quad (5.56)$$

where

ρ_0 = density at atmospheric conditions

p = gauge pressure, GPa

Therefore the general expression for the dimensionless density can be written as

$$\bar{\rho} = 1 + \frac{APE'}{1 + BPE'} \quad (5.57)$$

where A and B are constants dependent on the fluid.

5.6 Energy Equation

The equation used to determine the distribution of temperature within the fluid is a mathematical statement of the principle of energy conservation. As the lubricant is sheared, work is done on it, and there is a temperature rise that in turn changes the viscosity of the fluid. This variation of viscosity must be included in many solutions of the Reynolds equation, particularly when sliding occurs. Likewise from the standpoint of heat transfer and thermal distortion it is desirable to determine the temperature gradients of the energy equation for a fluid, and the full equation can be written as

$$\rho \frac{dE_a}{dt} = J^* \left[\frac{\partial}{\partial x} \left(\tilde{k} \frac{\partial T_m}{\partial x} \right) + \frac{\partial}{\partial y} \left(\tilde{k} \frac{\partial T_m}{\partial y} \right) + \frac{\partial}{\partial z} \left(\tilde{k} \frac{\partial T_m}{\partial z} \right) \right] - p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \gamma$$

Convection
Conduction
Adiabatic compression
Viscous dissipation

(5.58)

where

$$\begin{aligned} \gamma = \eta \left\{ \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right. \\ + 2 \left(\frac{\partial u}{\partial z} \frac{\partial w}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial v}{\partial z} \right) + \frac{4}{3} \left[\frac{\partial u}{\partial x} \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) \right. \\ \left. \left. + \frac{\partial v}{\partial y} \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) + \frac{\partial w}{\partial z} \left(\frac{\partial w}{\partial z} - \frac{\partial v}{\partial y} \right) \right] \right\} \end{aligned} \quad (5.59)$$

The convection term can usually be dropped from the energy equation in an analysis of elastohydrodynamic lubricating films. Likewise the rate at which heat can be conducted along the film (i.e., in the x and y directions) is small compared with the rate of conduction to the bearing surfaces. The significant terms in the viscous dissipation expression are seen to be those involving differentials of u and v in the z direction, and the reduced energy equation for lubricating films thus becomes

$$J^* \frac{\partial}{\partial z} \left(\tilde{k} \frac{\partial T_m}{\partial z} \right) - p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \eta \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right] = 0 \quad (5.60)$$

If the lubricant is considered to be an incompressible fluid, the adiabatic compression term in equation (5.60) is zero. Furthermore the thermal conductivity of liquid lubricants can often be treated as a constant in lubrication problems, and if both these assumptions are acceptable, the energy equation adopts the following form:

$$J^* \tilde{k} \frac{\partial^2 T_m}{\partial z^2} + \eta \left[\left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right] = 0 \quad (5.61)$$

This equation represents a balance between the rate of heat production within the film by viscous action and the rate of heat conduction from the film to the solids.

5.7 Elasticity Equation

The next task is to calculate the deformation in the ellipsoidal solids that represent an elastohydrodynamic contact as shown in Figure 2.18. Dowson (1965) distinguished between two modes of deformation that may exist in machine elements. In one mode the contact geometry may be affected by overall distortion of the elastic machine element resulting from applied loads, as shown in Figure 5.3(a). In the other the normal stress distribution in the vicinity of the contact zone may produce local elastic deformations that are significant when compared with the lubricant film thickness, as shown in Figure 5.3(b). This is

the mode of deformation with which this book is concerned. The important distinction is that the first form of deformation is relatively insensitive to the distribution and magnitude of the stresses in the contact zone, whereas the second mode of deformation is intimately linked to the local stress conditions.

The correct evaluation of elastic deformation on the surface of a solid depends on an adequate representation of the applied normal pressures. The simplest procedure is to divide the actual pressure distribution into rectangular blocks of uniform pressure and to permit each rectangle to be of such small dimensions that adequate predictions of elastic displacements ensue. More complex representations of two-dimensional pressure distributions within each rectangle would generally permit larger rectangles to be used, but the additional complexity of the expressions and the added computation times involved in numerical solutions make it desirable to exploit the simpler representation to the fullest.

The deformation analysis is developed in general form here since it is used in Chapter 7 in developing the elastohydrodynamic lubrication theory. The Dowson and Hamrock (1976) work is used extensively in this section.

In Chapter 3 the general geometry of two ellipsoidal solids in elastic contact is described. In the subsequent analysis it is convenient to consider the deformation of an equivalent elastic half-space subjected to a pressure distribution over the

ellipse of semimajor and semiminor axes a and b as previously defined in equations (3.13) and (3.14). The resulting elastic deformation can be considered to be equivalent to the total deformation of two elastic ellipsoids having elastic constants E_a, ν_a and E_b, ν_b , respectively, if the half-space is allocated the equivalent elastic parameter E' defined by equation (3.16).

Figure 5.4 shows a rectangular area of uniform pressure with the coordinate system to be used. From Timoshenko and Goodier (1951) the elastic deformation at a point (\bar{x}, \bar{y}) of a semi-infinite solid subjected to a pressure p at the point (\bar{x}_1, \bar{y}_1) can be written as

$$d\tilde{\delta} = \frac{2p \, d\bar{x}_1 \, d\bar{y}_1}{\pi E' \bar{r}}$$

The elastic deformation at a point (\bar{x}, \bar{y}) due to the uniform pressure over the rectangular area $2\bar{a} \times 2\bar{b}$ is thus

$$\tilde{\delta} = \frac{2P}{\pi} \int_{-\bar{a}}^{\bar{a}} \int_{-\bar{b}}^{\bar{b}} \frac{d\bar{x}_1 \, d\bar{y}_1}{\left[(\bar{y} - \bar{y}_1)^2 + (\bar{x} - \bar{x}_1)^2 \right]^{1/2}}$$

Integrating the preceding equation gives

$$\tilde{\delta} = \frac{2}{\pi} P \bar{D} \quad (5.62)$$

where

$$\begin{aligned}
\bar{D} = & (\bar{x} + \bar{b}) \ln \left[\frac{(\bar{y} + \bar{a}) + [(\bar{y} + \bar{a})^2 + (\bar{x} + \bar{b})^2]^{1/2}}{(\bar{y} - \bar{a}) + [(\bar{y} - \bar{a})^2 + (\bar{x} + \bar{b})^2]^{1/2}} \right] \\
& + (\bar{y} + \bar{a}) \ln \left[\frac{(\bar{x} + \bar{b}) + [(\bar{y} + \bar{a})^2 + (\bar{x} + \bar{b})^2]^{1/2}}{(\bar{x} - \bar{b}) + [(\bar{y} + \bar{a})^2 + (\bar{x} - \bar{b})^2]^{1/2}} \right] \\
& + (\bar{x} - \bar{b}) \ln \left[\frac{(\bar{y} - \bar{a}) + [(\bar{y} - \bar{a})^2 + (\bar{x} - \bar{b})^2]^{1/2}}{(\bar{y} + \bar{a}) + [(\bar{y} + \bar{a})^2 + (\bar{x} - \bar{b})^2]^{1/2}} \right] \\
& + (\bar{y} - \bar{a}) \ln \left[\frac{(\bar{x} - \bar{b}) + [(\bar{y} - \bar{a})^2 + (\bar{x} - \bar{b})^2]^{1/2}}{(\bar{x} + \bar{b}) + [(\bar{y} - \bar{a})^2 + (\bar{x} + \bar{b})^2]^{1/2}} \right]
\end{aligned}
\tag{5.63}$$

As a check on the validity of equation (5.62) the following two cases were evaluated:

Case 1: For $\bar{b} = \bar{a}$ and $\bar{x} = \bar{y} = 0$, equation (5.62) reduces to

$$\tilde{\delta} = \frac{16}{\pi} P \bar{a} \ln(1 + \sqrt{2})
\tag{5.64}$$

Equation (5.64) represents the elastic deformation at the center of the square of uniform pressure. This equation is in exact agreement with that presented by Timoshenko and Goodier (1951).

Case 2: For $\bar{b} = \bar{a}$ and $\bar{x} = \bar{y} = \bar{a}$, equation (5.62) reduces to

$$\bar{\delta} = \frac{8}{\pi} \bar{P} \bar{a} \ln(1 + \sqrt{2}) \quad (5.65)$$

Equation (5.65) represents the elastic deformation at the corner of a square of uniform pressure. This equation is also in agreement with the result presented by Timoshenko and Goodier (1951). From equations (5.64) and (5.65) we find the corner deformation to be one-half the deformation at the center of a square block of uniform pressure.

Now the term $\bar{\delta}$ in equation (5.62) represents the elastic deformation at a point (\bar{x}, \bar{y}) due to a rectangular area $2\bar{a} \times 2\bar{b}$ of uniform pressure p . If the contact ellipse is divided into a number of equal rectangular areas, the total deformation at a point (\bar{x}, \bar{y}) due to the contributions of the various rectangular areas of uniform pressure in the contact ellipse can be evaluated numerically. Figure 5.5 shows how the area inside and outside the contact ellipse can be divided into a number of equal rectangular areas. For illustration the contact is shown divided into a grid of 6 x 6 rectangular areas. The arrangement illustrated in Figure 5.5 can be used to evaluate the total elastic deformation, caused by the rectangular areas of uniform pressure within the contact ellipse, at any point inside or outside the contact ellipse as

$$\delta_{k,\ell} = \frac{2}{\pi} \sum_{j=1,2,\dots,6} \sum_{i=1,2,\dots,6} P_{i,j} \bar{D}_{m,s} \quad (5.66)$$

where

$$m = |k - i| + 1 \quad (5.67)$$

$$s = |l - j| + 1 \quad (5.68)$$

Note that $\bar{D}_{1,1}$ would be \bar{D} in equation (5.63) evaluated at $\bar{x} = 0, \bar{y} = 0$, while $\bar{D}_{2,3}$ would be evaluated at $\bar{x} = 2\bar{b}, \bar{y} = 4\bar{a}$.

The elastic deformation at the center of the rectangular area $\delta_{9,5}$ shown in Figure 5.5, caused by the pressure of the various rectangular areas in the contact ellipse, can be written as

$$\begin{aligned} \delta_{9,5} = \frac{2}{\pi} \left\{ P_{1,1} \bar{D}_{9,5} + P_{2,1} \bar{D}_{8,5} + \cdots + P_{6,1} \bar{D}_{4,5} \right. \\ + P_{1,2} \bar{D}_{9,4} + P_{2,2} \bar{D}_{8,4} + \cdots + P_{6,2} \bar{D}_{4,4} \\ + P_{1,3} \bar{D}_{9,3} + P_{2,3} \bar{D}_{8,3} + \cdots + P_{6,3} \bar{D}_{4,3} \\ + P_{1,4} \bar{D}_{9,2} + P_{2,4} \bar{D}_{8,2} + \cdots + P_{6,4} \bar{D}_{4,2} \left. \right\} \quad (5.69) \end{aligned}$$

Equation (5.69) points out more explicitly the meaning of equation (5.66). A numerical analysis of the elastic deformation of a contacting ellipsoidal solid and plane was performed by Dowson and Hamrock (1976). The analysis assumed that the pressure in the contact zone was Hertzian. It also assumed that the contact zone could be divided into rectangular areas with uniform pressure within each rectangular area. The resulting equations were programmed on a digital computer. Four limiting conditions were evaluated. They consisted of two extremes of applied normal load: a light load of 8.964 N (2 lbf), and a

heavy load of 896.4 N (200 lbf). The two other extremes are of the curvature of the contacting solids: two equal spheres in contact, and a ball and outer race of a ball bearing. It was speculated that conclusions drawn from the results obtained for these limiting conditions could also be made for any intermediate condition.

Figures 5.6, 5.7, and 5.8 illustrate the results of an investigation of the effects of mesh size on the accuracy of the calculations of elastic deformation presented by Dowson and Hamrock (1976). In these figures the solid curve represents the case of equal spheres in contact in which $R_x = R_y = 0.005558$ m (0.2188 in.), and the dashed curve represents the contact between a ball and outer race in a bearing in which $R_x = 0.0128$ m (0.5055 in.) and by $R_y = 0.150$ m (5.91 in.).

Figure 5.6 shows how the percentage difference in elastic deformation varies along the semimajor axis when $\bar{m} = 3, 4$, and 5 - the predictions being related to more exact solutions for three times as many divisions of the axes in each case (i.e., $\bar{m} = 9, 12$, and 15, respectively). In this figure, we see a large drop in the error term $100 [(\delta_{\bar{m}} - \delta_{3\bar{m}})/\delta_{3\bar{m}}]$ when going from $\bar{m} = 4$ to $\bar{m} = 5$. It was therefore argued that a value of \bar{m} of 5 would yield acceptable accuracy with minimum computing effort under most conditions.

The ratio of the elastic deformation to the distance separating the two undeformed solids in contact is shown as a func-

tion of the distance along the semimajor and semiminor axes in Figures 5.7 and 5.8, respectively. These figures show that the elastic deformation ceases to be a significant component of the overall separation of the solids at modest distances away from the edge of the Hertzian contact zone. To be more specific, from the curves shown in Figures 5.7 and 5.8 we see that for equal spheres in contact, represented by solid lines in the figures, $\delta/S < 0.05$ corresponds to $x > 2.6 b$ and $y > 2.6 a$. This means that the elastic deformation is less than 5 percent of the film thickness due to the geometry of the undeformed solids at distances of only 1.6 times the semiaxes dimensions beyond the edge of the Hertzian contact zone. For the ball-and-outer-race example a value of $\delta/S < 0.05$ is achieved when $y > 1.9 a$ and $x > 4.0 b$. This means that the elastic deformation is less than 5 percent of the film shape due to the undeformed geometry at distances beyond the contours of the Hertzian ellipse of only 0.9 and 3.0 times the semimajor and semiminor axes, respectively.

The great simplicity introduced into the calculation of elastic deformations by the assumption that the pressure is uniform over rectangular blocks has already been mentioned. This assumption enabled the wide range of approximate numerical solutions required to generate the empirical film thickness equations to be obtained in an acceptable time and with a justifiable computing effort. Evans and Snidle (1978) have discussed a

refinement to the numerical procedure based largely on a more detailed approach to the calculation of elastic deformations. In this approach the elastic deformation at any point resulting from a block of pressure in region A is determined by dividing A into two regions, B and C, and adopting different procedures for the evaluation of the basic integrals in the two regions. In the central region, C, where a singularity occurs in a straightforward approach to the problem, the pressure function is approximated by a biquadratic polynomial to facilitate numerical analysis. In the remaining region, B, the integral is evaluated in a straightforward manner by the application of Simpson's rule.

Evans and Snidle (1978) provided two full solutions to the elastohydrodynamic problem for nominal point or circular contacts ($k = 1$) that revealed most clearly all the characteristics of film shape and pressure distribution associated with the problem. They also investigated the influence of lubricant starvation, as determined by the location of the inlet boundary and the location of the side boundary to the computing zone, on film thickness. They concluded that both factors could play an important part in determining the accuracy of numerical solutions, and both factors are considered in detail in the numerical solutions presented in this text.

A comparison between the predictions of the film thickness equations developed in this text, which are based on solutions of the elasticity equation as outlined in this chapter, and the

refined approach presented for only two conditions by Evans and Snidle (1978) provides justifiable confidence in the adequacy of the numerical procedures adopted. In one particular case, corresponding to a sphere of radius 0.025 m rolling near a plane with a velocity of 0.5 m/s while supporting a load of 120 N, the central and minimum film thicknesses calculated by Evans and Snidle (1978) were predicted to within 0.3 and 1.4 percent, respectively, by equations (8.41) and (8.23) in this text. We are grateful to Dr. Snidle (private communication, April 1980) for the information that enabled this comparison to be made.

5.8 Closure

This chapter has dealt with the basic lubrication and elasticity equations used in the study of hydrodynamic lubrication. The Navier-Stokes equations have been derived in general form and then reduced within the framework of assumptions that are deemed to be valid for lubricating films. The Reynolds equation was then derived in two different ways: one from the coupling of the reduced Navier-Stokes equations with the continuity equation, the other from first principles and the direct combination of well-established viscous flow expressions with a statement of flow continuity. The Reynolds equation contains viscosity and density as parameters. These properties depend on the temperature and pressure, and it is therefore necessary to establish

expressions relating these quantities. The energy equation, which governs the temperature distribution within the lubricant, is also presented in this chapter. It is noted that solutions to both the energy and Reynolds equations may be required in some situations.

The Reynolds equation also contains the film thickness as a parameter. The film thickness is a function of the undeformed geometry and of the elastic behavior of the bearing surfaces. The elasticity equation has therefore been developed in this chapter. The coupling of the Reynolds and elasticity equations will be dealt with in Chapter 7, where the full theory of elastohydrodynamic lubrication is developed.

SYMBOLS

| | |
|---------------------------------------|--|
| A | constant used in equation (3.113) |
| $A^*, B^*, C^*,$ D^*, L^*, M^* } | relaxation coefficients |
| A_v | drag area of ball, m^2 |
| a | semimajor axis of contact ellipse, m |
| \bar{a} | $a/2\bar{m}$ |
| B | total conformity of bearing |
| b | semiminor axis of contact ellipse, m |
| \bar{b} | $b/2\bar{m}$ |
| C | dynamic load capacity, N |
| C_v | drag coefficient |
| C_1, \dots, C_8 | constants |
| c | $19,609 \text{ N/cm}^2$ ($28,440 \text{ lbf/in}^2$) |
| \bar{c} | number of equal divisions of semimajor axis |
| D | distance between race curvature centers, m |
| \tilde{D} | material factor |
| \bar{D} | defined by equation (5.63) |
| De | Deborah number |
| d | ball diameter, m |
| \bar{d} | number of divisions in semiminor axis |
| d_a | overall diameter of bearing (Figure 2.13), m |
| d_b | bore diameter, m |
| d_e | pitch diameter, m |
| d'_e | pitch diameter after dynamic effects have acted on ball, m |
| d_i | inner-race diameter, m |
| d_o | outer-race diameter, m |

| | |
|--------------------------|---|
| E | modulus of elasticity, N/m^2 |
| E' | effective elastic modulus, $2 / \left(\frac{1 - \nu_a^2}{E_a} + \frac{1 - \nu_b^2}{E_b} \right)$, N/m^2 |
| E_a | internal energy, m^2/s^2 |
| \tilde{E} | processing factor |
| E_1 | $[(\tilde{H}_{\min} - H_{\min})/H_{\min}] \times 100$ |
| \mathcal{E} | elliptic integral of second kind with modulus $(1 - 1/k^2)^{1/2}$ |
| $\overline{\mathcal{E}}$ | approximate elliptic integral of second kind |
| e | dispersion exponent |
| F | normal applied load, N |
| F^* | normal applied load per unit length, N/m |
| \tilde{F} | lubrication factor |
| \overline{F} | integrated normal applied load, N |
| F_c | centrifugal force, N |
| F_{\max} | maximum normal applied load (at $\psi = 0$), N |
| F_r | applied radial load, N |
| F_t | applied thrust load, N |
| F_ψ | normal applied load at angle ψ , N |
| \mathcal{F} | elliptic integral of first kind with modulus $(1 - 1/k^2)^{1/2}$ |
| $\overline{\mathcal{F}}$ | approximate elliptic integral of first kind |
| f | race conformity ratio |
| f_b | rms surface finish of ball, m |
| f_r | rms surface finish of race, m |
| G | dimensionless materials parameter, αE |
| G^* | fluid shear modulus, N/m^2 |
| \tilde{G} | hardness factor |
| g | gravitational constant, m/s^2 |

| | |
|-------------------|---|
| g_E | dimensionless elasticity parameter, $w^{8/3}/U^2$ |
| g_V | dimensionless viscosity parameter, GW^3/U^2 |
| H | dimensionless film thickness, h/R_x |
| \hat{H} | dimensionless film thickness, $H(W/U)^2 = F^2 h/u^2 n_0^2 R_x^3$ |
| H_C | dimensionless central film thickness, h_C/R_x |
| $H_{C,s}$ | dimensionless central film thickness for starved lubrication condition |
| H_f | frictional heat, N m/s |
| H_{min} | dimensionless minimum film thickness obtained from EHL elliptical-contact theory |
| $H_{min,r}$ | dimensionless minimum film thickness for a rectangular contact |
| $H_{min,s}$ | dimensionless minimum film thickness for starved lubrication condition |
| \tilde{H}_C | dimensionless central film thickness obtained from least-squares fit of data |
| \tilde{H}_{min} | dimensionless minimum film thickness obtained from least-squares fit of data |
| \bar{H}_C | dimensionless central-film-thickness - speed parameter, $H_C U^{-0.5}$ |
| \bar{H}_{min} | dimensionless minimum-film-thickness - speed parameter, $H_{min} U^{-0.5}$ |
| \bar{H}_0 | new estimate of constant in film thickness equation |
| h | film thickness, m |
| h_C | central film thickness, m |
| h_i | inlet film thickness, m |

| | |
|-------------------|---|
| h_m | film thickness at point of maximum pressure, where $dp/dx = 0$, m |
| h_{min} | minimum film thickness, m |
| h_0 | constant, m |
| I_d | diametral interference, m |
| I_p | ball mass moment of inertia, $m \text{ N s}^2$ |
| I_r | integral defined by equation (3.76) |
| I_t | integral defined by equation (3.75) |
| J | function of k defined by equation (3.8) |
| J^* | mechanical equivalent of heat |
| \bar{J} | polar moment of inertia, $m \text{ N s}^2$ |
| K | load-deflection constant |
| k | ellipticity parameter, a/b |
| \bar{k} | approximate ellipticity parameter |
| \tilde{k} | thermal conductivity, $\text{N/s } ^\circ\text{C}$ |
| k_f | lubricant thermal conductivity, $\text{N/s } ^\circ\text{C}$ |
| L | fatigue life |
| L_a | adjusted fatigue life |
| L_t | reduced hydrodynamic lift, from equation (6.21) |
| L_1, \dots, L_4 | lengths defined in Figure 3.11, m |
| L_{10} | fatigue life where 90 percent of bearing population will endure |
| L_{50} | fatigue life where 50 percent of bearing population will endure |
| ℓ | bearing length, m |
| $\bar{\ell}$ | constant used to determine width of side-leakage region |
| M | moment, Nm |

| | |
|-------------|---|
| M_g | gyroscopic moment, Nm |
| M_p | dimensionless load-speed parameter, $WU^{-0.75}$ |
| M_s | torque required to produce spin, N m |
| m | mass of ball, $N\ s^2/m$ |
| m^* | dimensionless inlet distance at boundary between fully flooded and starved conditions |
| \tilde{m} | dimensionless inlet distance (Figures 7.1 and 9.1) |
| \bar{m} | number of divisions of semimajor or semiminor axis |
| m_w | dimensionless inlet distance boundary as obtained from Wedeven, et al. (1971) |
| N | rotational speed, rpm |
| n | number of balls |
| n^* | refractive index |
| \bar{n} | constant used to determine length of outlet region |
| p | dimensionless pressure |
| p_D | dimensionless pressure difference |
| p_d | diametral clearance, m |
| p_e | free endplay, m |
| p_{Hz} | dimensionless Hertzian pressure, N/m^2 |
| p | pressure, N/m^2 |
| p_{max} | maximum pressure within contact, $3F/2\pi ab$, N/m^2 |
| $p_{iv,as}$ | isoviscous asymptotic pressure, N/m^2 |
| Q | solution to homogeneous Reynolds equation |
| Q_m | thermal loading parameter |
| \bar{Q} | dimensionless mass flow rate per unit width, $q\eta_0/\rho_0 E'R^2$ |
| q_f | reduced pressure parameter |
| q_x | volume flow rate per unit width in x direction, m^2/s |

| | |
|----------------------------------|--|
| q_y | volume flow rate per unit width in y direction, m^2/s |
| R | curvature sum, m |
| R_a | arithmetical mean deviation defined in equation (4.1), m |
| R_c | operational hardness of bearing material |
| R_x | effective radius in x direction, m |
| R_y | effective radius in y direction, m |
| r | race curvature radius, m |
| $r_{ax}, r_{bx}, r_{ay}, r_{by}$ | radii of curvature, m |
| r_c, ϕ_c, z | cylindrical polar coordinates |
| r_s, θ_s, ϕ_s | spherical polar coordinates |
| \bar{r} | defined in Figure 5.4 |
| S | geometric separation, m |
| S^* | geometric separation for line contact, m |
| S_0 | empirical constant |
| s | shoulder height, m |
| T | τ_0/p_{max} |
| \tilde{T} | tangential (traction) force, N |
| T_m | temperature, $^{\circ}C$ |
| T_b^* | ball surface temperature, $^{\circ}C$ |
| T_f^* | average lubricant temperature, $^{\circ}C$ |
| ΔT^* | ball surface temperature rise, $^{\circ}C$ |
| T_1 | $(\tau_0/p_{max})_{k=1}$ |
| T_v | viscous drag force, N |
| t | time, s |
| t_a | auxiliary parameter |
| u_B | velocity of ball-race contact, m/s |

| | |
|---|--|
| u_c | velocity of ball center, m/s |
| U | dimensionless speed parameter, $\eta_0 u / E' R_x$ |
| u | surface velocity in direction of motion, $(u_a + u_b)/2$, m/s |
| \bar{u} | number of stress cycles per revolution |
| Δu | sliding velocity, $u_a - u_b$, m/s |
| v | surface velocity in transverse direction, m/s |
| W | dimensionless load parameter, $F/E'R^2$ |
| w | surface velocity in direction of film, m/s |
| x | dimensionless coordinate, x/R_x |
| y | dimensionless coordinate, y/R_x |
| x_t, y_t | dimensionless grouping from equation (6.14) |
| x_a, y_a, z_a | external forces, N |
| Z | constant defined by equation (3.48) |
| z_1 | viscosity pressure index, a dimensionless constant |
| $\left. \begin{array}{l} x, \tilde{x}, \bar{x}, \bar{x}_1 \\ y, \tilde{y}, \bar{y}, \bar{y}_1 \\ z, \tilde{z}, \bar{z}, \bar{z}_1 \end{array} \right\}$ | coordinate system |
| α | pressure-viscosity coefficient of lubrication, m^2/N |
| α_a | radius ratio, R_y/R_x |
| β | contact angle, rad |
| β_f | free or initial contact angle, rad |
| β' | iterated value of contact angle, rad |
| r | curvature difference |
| γ | viscous dissipation, $N/m^2 \cdot s$ |
| $\dot{\gamma}$ | total strain rate, s^{-1} |
| $\dot{\gamma}_e$ | elastic strain rate, s^{-1} |
| $\dot{\gamma}_v$ | viscous strain rate, s^{-1} |

| | |
|------------------|--|
| γ_a | flow angle, deg |
| δ | total elastic deformation, m |
| δ^* | lubricant viscosity temperature coefficient, $^{\circ}\text{C}^{-1}$ |
| δ_D | elastic deformation due to pressure difference, m |
| δ_r | radial displacement, m |
| δ_t | axial displacement, m |
| δ_x | displacement at some location x , m |
| $\bar{\delta}$ | approximate elastic deformation, m |
| $\tilde{\delta}$ | elastic deformation of rectangular area, m |
| ϵ | coefficient of determination |
| ϵ_1 | strain in axial direction |
| ϵ_2 | strain in transverse direction |
| ζ | angle between ball rotational axis and bearing centerline (Figure 3.10) |
| ζ_a | probability of survival |
| η | absolute viscosity at gauge pressure, N s/m^2 |
| $\bar{\eta}$ | dimensionless viscosity, η/η_0 |
| η_0 | viscosity at atmospheric pressure, N s/m^2 |
| η_{∞} | $6.31 \times 10^{-5} \text{ N s/m}^2$ (0.0631 cP) |
| θ | angle used to define shoulder height |
| Λ | film parameter (ratio of film thickness to composite surface roughness) |
| λ | equals 1 for outer-race control and 0 for inner-race control |
| λ_a | second coefficient of viscosity |
| λ_b | Archard-Cowking side-leakage factor, $(1 + 2/3 \alpha_a)^{-1}$ |
| λ_c | relaxation factor |

| | |
|------------------|---|
| μ | coefficient of sliding friction |
| μ^* | $\bar{\rho}/\bar{\eta}$ |
| ν | Poisson's ratio |
| ξ | divergence of velocity vector, $(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z)$, s^{-1} |
| ρ | lubricant density, $N\ s^2/m^4$ |
| $\bar{\rho}$ | dimensionless density, ρ/ρ_0 |
| ρ_0 | density at atmospheric pressure, $N\ s^2/m^4$ |
| σ | normal stress, N/m^2 |
| σ_1 | stress in axial direction, N/m^2 |
| τ | shear stress, N/m^2 |
| τ_0 | maximum subsurface shear stress, N/m^2 |
| $\tilde{\tau}$ | shear stress, N/m^2 |
| $\tilde{\tau}_e$ | equivalent stress, N/m^2 |
| $\tilde{\tau}_L$ | limiting shear stress, N/m^2 |
| Φ | ratio of depth of maximum shear stress to semiminor axis of contact ellipse |
| Φ^* | $\rho H^{3/2}$ |
| Φ_1 | $(\Phi)_{k=1}$ |
| ϕ | auxiliary angle |
| ϕ_T | thermal reduction factor |
| ψ | angular location |
| ψ_ℓ | limiting value of ψ |
| Ω_i | absolute angular velocity of inner race, rad/s |
| Ω_0 | absolute angular velocity of outer race, rad/s |
| ω | angular velocity, rad/s |
| ω_B | angular velocity of ball-race contact, rad/s |
| ω_b | angular velocity of ball about its own center, rad/s |

ω_c angular velocity of ball around shaft center, rad/s

ω_s ball spin rotational velocity, rad/s

Subscripts:

a solid a

b solid b

c central

bc ball center

IE isoviscous-elastic regime

IR isoviscous-rigid regime

i inner race

K Kapitza

min minimum

n iteration

o outer race

PVE piezoviscous-elastic regime

PVR piezoviscous-rigid regime

r for rectangular area

s for starved conditions

x,y,z coordinate system

Superscript:

(—) approximate

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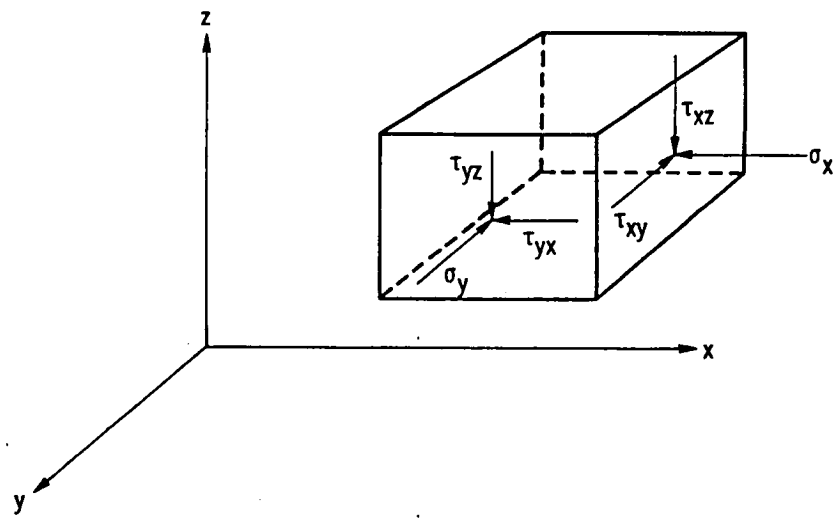


Figure 5.1. - Stresses on two surfaces of fluid element.

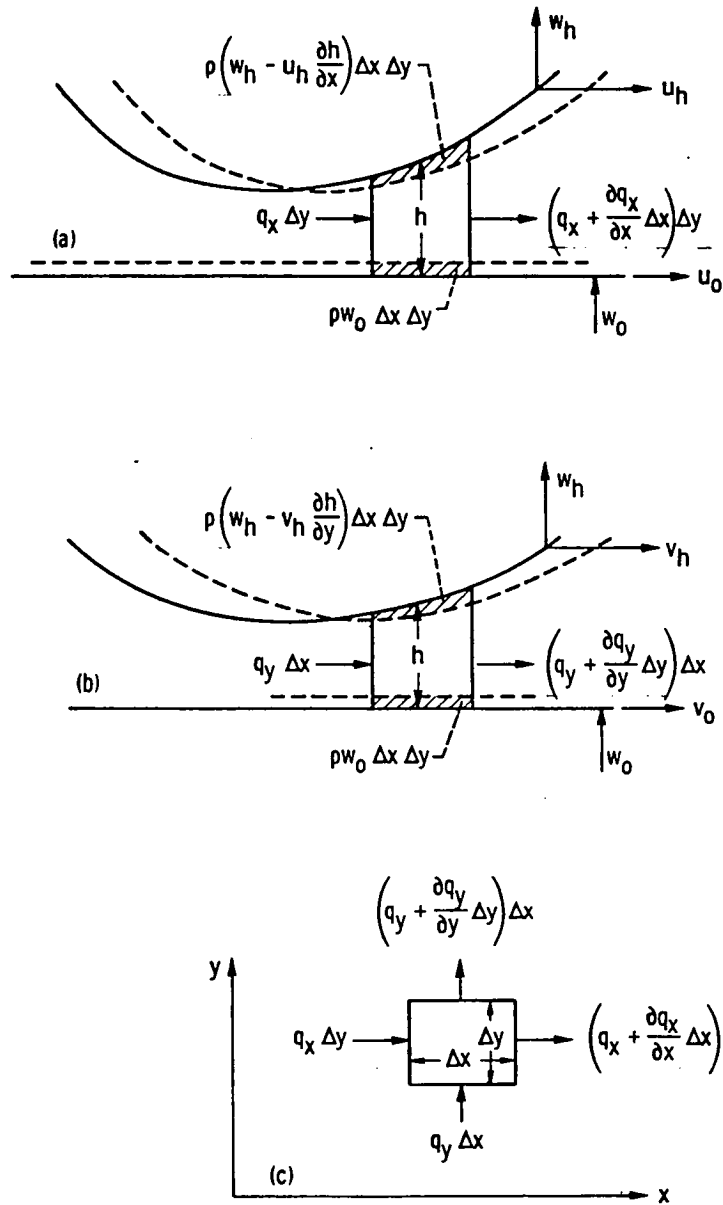


Figure 5.2 - Mass flow through rectangular-section control volume.
(a) x, z plane. (b) y, z plane. (c) x, y plane.

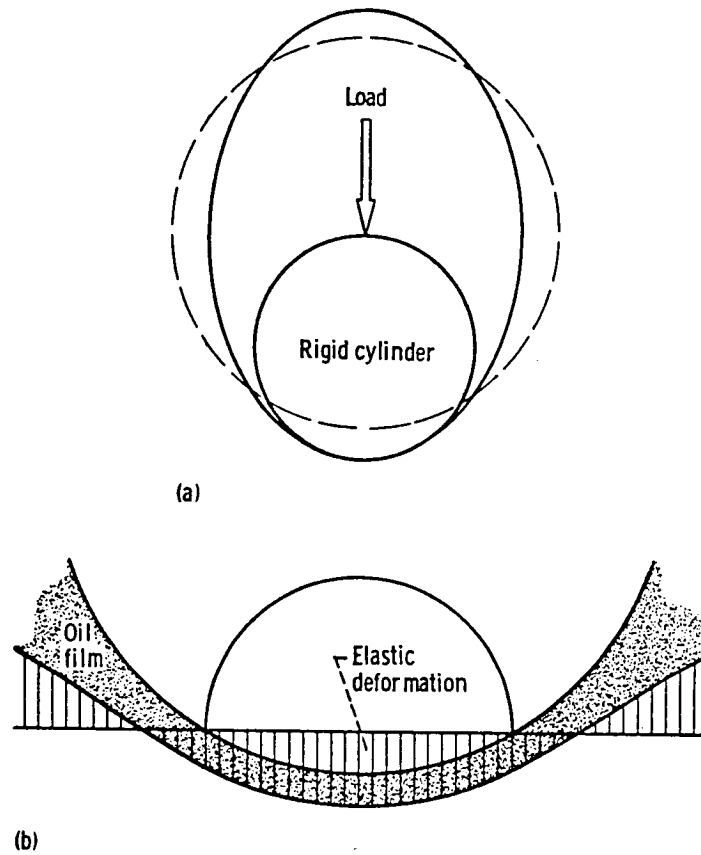


Figure 5.3. - Types of elastic deformation. (a) Distortion of element. (b) Local elastic deformation.

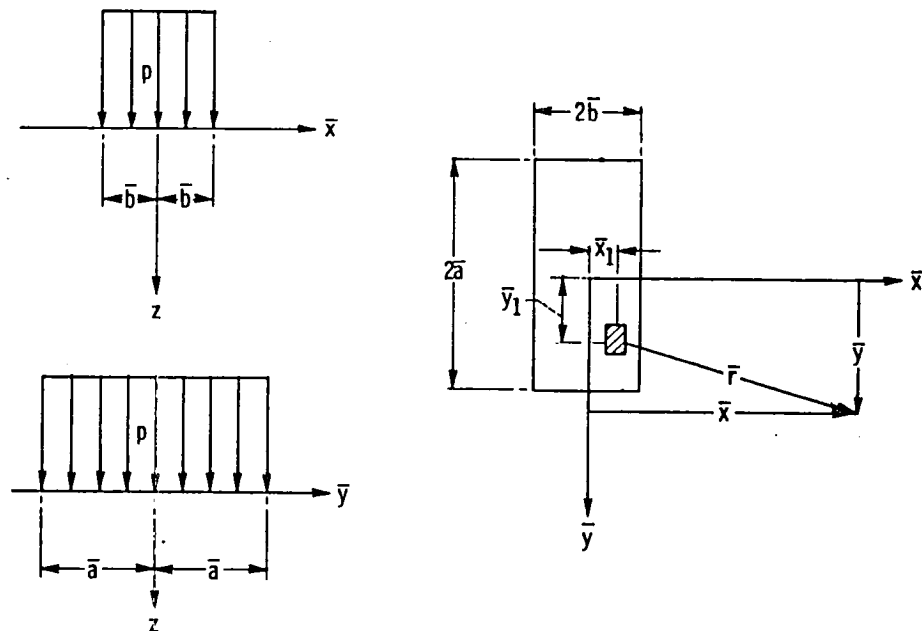


Figure 5.4. - Surface deformation of semi-infinite body subjected to uniform pressure over rectangular area.

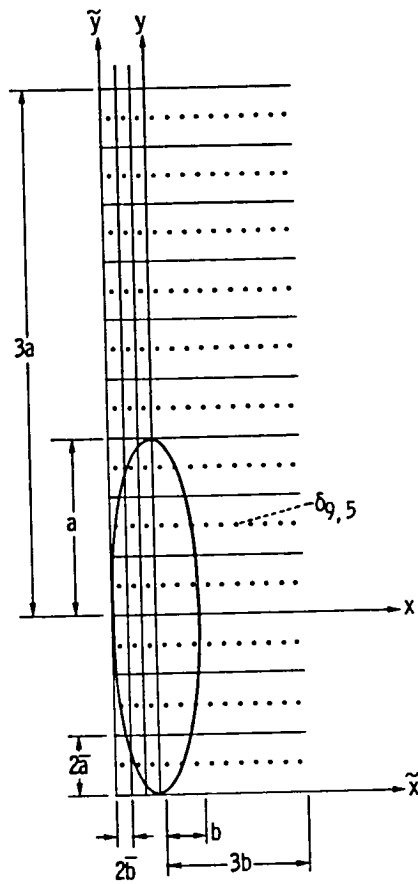


Figure 5.5. - Division of area in and around contact into equal rectangular areas.

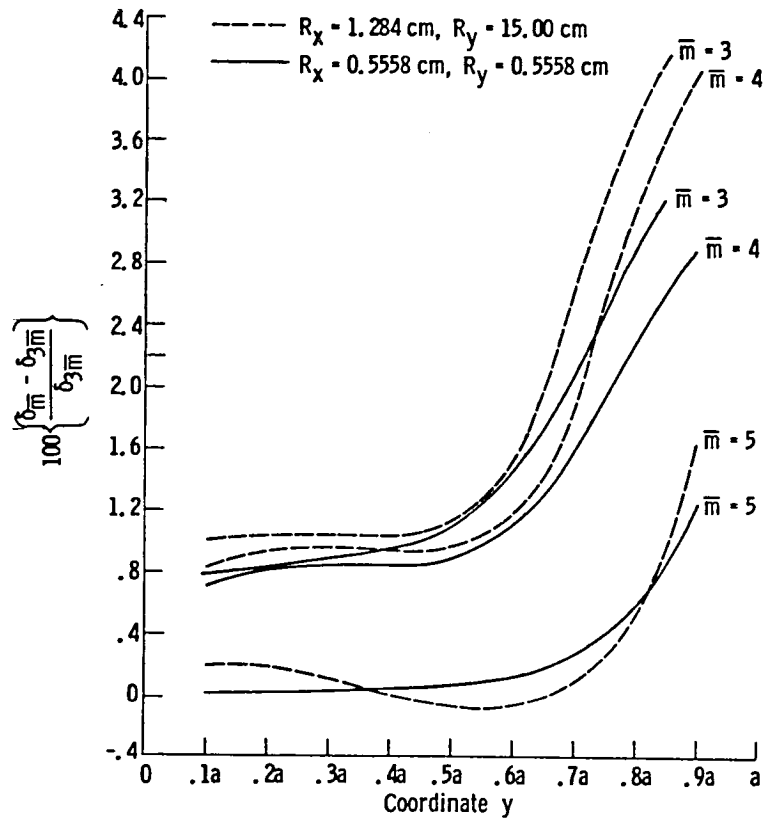


Figure 5.6. - Effect of location along semimajor axis on percentage difference between elastic deformation when $\bar{m} = 3, 4$, and 5 and the more exact film shape when $\bar{m} = 9, 12$, and 15, respectively.

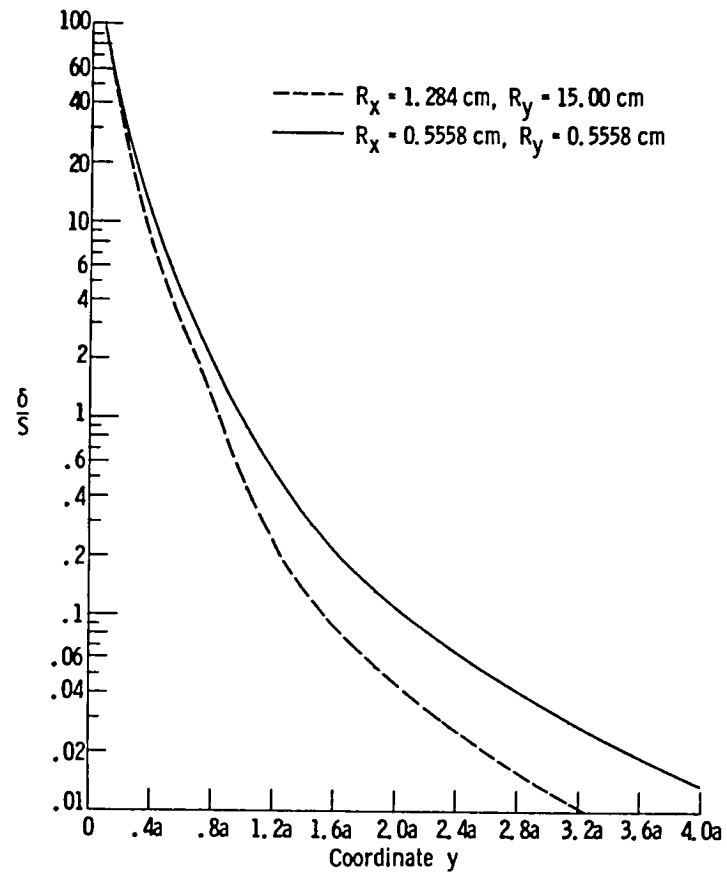


Figure 5.7. - Effect of location along semimajor axis on ratio of elastic deformation to distance separating two solids in contact due to geometry of the solids for $\bar{m} = 5$.

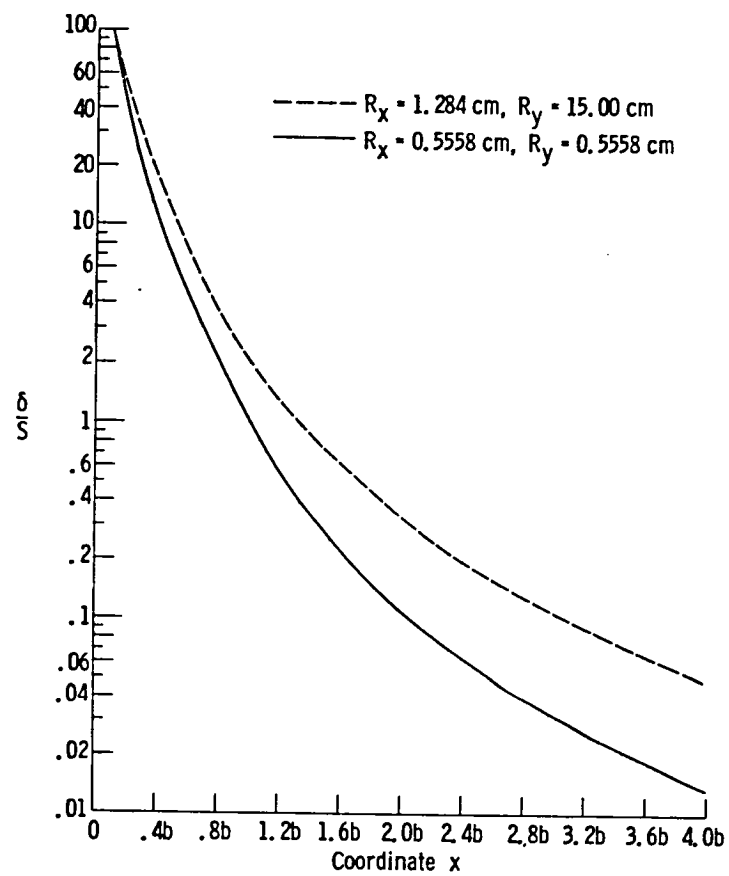


Figure 5.8. - Effect of location along semiminor axis on ratio of elastic deformation to distance separating two solids in contact due to geometry of the solids for $\bar{m} = 5$.

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| 1. Report No. NASA TM-81693 | | 2. Government Accession No. | | 3. Recipient's Catalog No. | |
| 4. Title and Subtitle BASIC LUBRICATION EQUATIONS | | | | 5. Report Date December 1981 | |
| | | | | 6. Performing Organization Code 505-32-42 | |
| 7. Author(s) Bernard J. Hamrock and Duncan Dowson | | | | 8. Performing Organization Report No. E-209 | |
| 9. Performing Organization Name and Address National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135 | | | | 10. Work Unit No. | |
| | | | | 11. Contract or Grant No. | |
| 12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546 | | | | 13. Type of Report and Period Covered Technical Memorandum | |
| | | | | 14. Sponsoring Agency Code | |
| 15. Supplementary Notes Bernard J. Hamrock, NASA Lewis Research Center, Cleveland, Ohio, and Duncan Dowson, Institute of Tribology, Department of Mechanical Engineering, The University of Leeds, Leeds, England. Published as Chapter 5 in Ball Bearing Lubrication by Bernard J. Hamrock and Duncan Dowson, John Wiley & Sons, Inc., September 1981. | | | | | |
| 16. Abstract <p>Lubricants are usually Newtonian fluids, in which the rate of shear is linearly related to the shear stress. The fluid is normally assumed to experience laminar flow, and the basic equations used to describe the flow are the Navier-Stokes equations of motion. The Navier-Stokes equations are derived as simply as possible. The study of hydrodynamic lubrication is, from a mathematical standpoint, the application of a reduced form of these Navier-Stokes equations in association with the continuity equation. The Reynolds equation can also be derived from first principles, provided of course that the same basic assumptions are adopted in each case. Both methods are used in deriving the Reynolds equation, and the assumptions inherent in reducing the Navier-Stokes equations are specified. The Reynolds equation contains viscosity and density terms. These properties of the lubricant depend on temperature and pressure; hence it is often necessary to couple the Reynolds equation with the energy equation. Therefore the lubricant properties and the energy equation are presented. The Reynolds equation also contains the film thickness as a parameter. The film thickness is a function of the elastic behavior of the bearing surface, and the governing elasticity equation is therefore presented.</p> | | | | | |
| 17. Key Words (Suggested by Author(s)) Newtonian fluids; Navier-Stokes equations; Reynolds equation; Lubricant properties; Energy equation; Governing elasticity equation | | | 18. Distribution Statement Unclassified - unlimited STAR Category 37 | | |
| 19. Security Classif. (of this report) Unclassified | | 20. Security Classif. (of this page) Unclassified | | 21. No. of Pages | |
| | | | | 22. Price* | |

National Aeronautics and
Space Administration

Washington, D.C.
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